

# Making Music with Mathematics

Have you ever wondered how to bring music into your mathematics lessons? Music can be a rich context for facilitating mathematical connections. However, for those of us who are not musically inclined, encouraging mathematical connections through music can seem overwhelming. The following tasks are a way to bring music into your classroom with rewarding results. I have used these mathematics tasks with high school students, as well with as preservice and in-service teachers. Students become engaged in problem-solving and reasoning tasks as they determine how to fill bottles with water to create musical notes by blowing across the tops of the bottles and as they create mathematical models that represent these notes. In addition to discussing the tasks, this article presents examples of students' work from these experiences and describes mathematics and physics concepts related to these tasks.

## PROBLEM: PLAYING A SONG

A good way to introduce music into a mathematics class is by challenging the students to play "Mary Had a Little Lamb"—or "La Cucaracha," a familiar Spanish tune—with 20-ounce plastic soft-drink bottles filled with water. I have adapted

this task from *CBL System Experiment Workbook* (Texas Instruments 1994). The students typically suggest that the bottles can be filled with different amounts of water to make different notes; however, they wonder how they can be sure that they have the correct notes unless they have a perfect ear for pitch. At that point, I introduce the technology that we will use to determine the notes: a Texas Instruments graphing calculator, a Calculator-Based Laboratory (CBL), and a CBL microphone. Students use the CBL to collect sound-data information from the microphone and store it in the graphing calculator (see **photograph**).

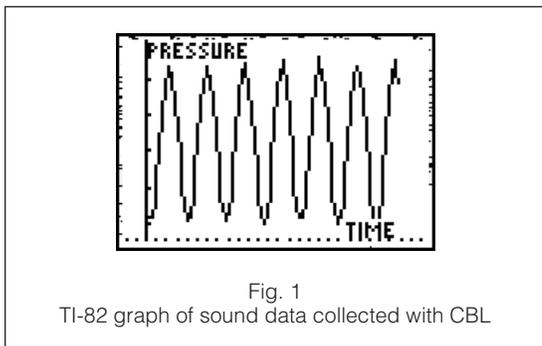
A graphing-calculator program drives the CBL collection of data. As a student creates a constant

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*Maria Fernandez, mariaf@u.arizona.edu, teaches and conducts research at the University of Arizona, Tucson, AZ 85721. Among her interests are using technology to teach mathematics. She thanks Cecilia Taylor and her Palo Verde High School students for their cooperation.*

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tone by blowing across the top of a partially filled bottle, the CBL microphone is held next to the bottle, and **TUNE**, the calculator program that drives the data collection, is executed. A student who has difficulty creating a constant tone can use a drinking straw to blow across the top of the bottle. **Figure 1** shows the graphical representation of sound-wave data collected with the **TUNE** program. **TUNE** graphs the data points using a line graph, thereby giving the graph a jagged appearance.



Like playing a flute or piccolo, blowing across the opening at the top of the bottle creates the Bernoulli effect, which sucks air from above the edge of the bottle into the bottle, thus disturbing the air molecules inside and producing excess air that tries to return to the outside (Moravcsik 1987). As the air pushes above the edge, the process repeats, producing an airstream that oscillates rapidly above and below the edge of the bottle. This process produces regions of higher-than-normal and lower-than-normal pressure in the bottle, forming the sinusoidal shape of the pressure-versus-time graph in **figure 1**. Each musical note is associated with a specific frequency of changes in pressure, or *cycles*. The frequency of a sound wave is measured in hertz, or cycles per second, for example, 262 Hz = 262 cycles per second.

**Table 1** shows the frequencies of musical notes beginning at middle C in the scale of equal temperament (Olson 1967). In this scale the octave is divided into twelve equal intervals; the higher the note, the greater the frequency. The C notes in **table 1** reveal the doubling of the frequency of corresponding notes in successive octaves. Reinthaler (1990) described this doubling and other ratios found in the Pythagorean and equal-temperament musical scales. In addition to comparing these scales, mathematics students can investigate the ratios of the musical notes in these scales.

### CREATING MUSICAL NOTES

After the teacher introduces the task, students are assigned to small groups. Each group is given one note from the song and receives the equipment necessary to create its note—

Note	Frequency in Hz
C	261.626
C# D <sup>b</sup>	277.183
D	293.665
D# E <sup>b</sup>	311.127
E	329.628
F	349.228
F# G <sup>b</sup>	369.994
G	391.995
G# A <sup>b</sup>	415.305
A	440
A# B <sup>b</sup>	466.164
B	493.883
next octave	
C	523.251

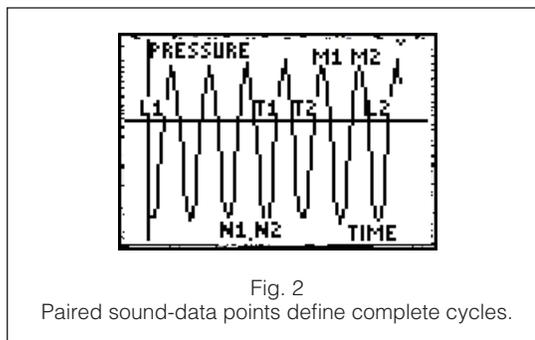
- two plastic soft-drink bottles, one for the note and one to add or remove water;
- a CBL;
- a CBL microphone; and
- two calculators, one TI-82 to use with the CBL and one for calculations.

If necessary, groups can share the CBL and CBL microphone.

Creating the musical note involves students in mathematical problem solving as they devise and implement a plan to create their note. Students begin by experimenting with the equipment. After they collect the sinusoidal sound data, they discuss strategies for determining their frequencies. These strategies involve proportional reasoning. The students are not told that the relationship frequency equals 1/period. They must determine the frequency of their note on the basis of an initial introduction to frequency and hertz.

Student groups have selected different methods for determining the frequency of their sound data. Some groups used the length of one cycle, or period, to determine the frequency of the sound wave; others used the length of several complete cycles. **Figure 2** is a pressure-versus-time graph showing different ways in which students defined one cycle: the part of the curve between two adjacent maximum points, for example, **M1** and **M2**; two adjacent

*Challenge students to play “La Cucaracha” with water-filled soft-drink bottles*



minimum points, for example, **N1** and **N2**; or two adjacent points of increase or points of decrease along the same horizontal line, for example, **T1** and **T2**. Students found the length, in seconds, of one cycle of their curve using the  $x$ -values of their two points, for example,  $x_2 - x_1$ , and used this length to determine the frequency of the sound wave.

Some groups developed a proportion, such as

$$\frac{1 \text{ cycle}}{(x_2 - x_1) \text{ sec.}} = \frac{f \text{ cycles}}{1 \text{ sec.}},$$

where  $(x_2 - x_1)$  is the length of one cycle, to find the frequency,  $f$ , of their curve (see **fig. 3**). Others divided one cycle by  $(x_2 - x_1)$  seconds to determine the frequency, that is, cycles per second, of their curve.

We began by obtaining base line data of how much water was already in the bottle. After recording the sound on the calculator, we measured the duration of one cycle. Taking the one cycle, we figured out how many cycles would occur in one second. (See the formula below) | .0040896 - .0078496 |

$$\frac{\text{Cycle}}{\text{second}} = \frac{1}{.00376} = \frac{x}{1}$$

↑ duration of one cycle      ↑ one full second

$$1 = .00376x$$

$$266 \approx x$$

$$\approx 266 \text{ Hz}$$

The above is what worked. In order to obtain the correct answer, we adjusted the water according to the #z we found. Too high, we poured water out; too low, we added more water. We had C# forever! 13 trials later we are close enough for Congress.

Fig. 3  
Student explanation of note using only one cycle

Rather than use only one cycle, some student groups used the length in seconds of all the complete cycles of their sound data to determine the frequency of their sound wave, for example, **L1** to **L2** in **figure 2**. This method is more accurate because the frequency is based on all the cycles in the graph, thus on the average length of a cycle, rather than on the measure of only one cycle. Some

of these groups divided the number of complete cycles by their length in seconds, for example,

$$\frac{6 \text{ cycles}}{(x_2 - x_1) \text{ sec.}}$$

The resulting quotient is the number of cycles per second, the frequency of the note.

**Figure 4** presents another solution that is based on all the complete cycles in the data. The first time that a group of students explained this approach to me, I was surprised by its complexity and had to think carefully about its appropriateness. First, they found the length, in fractions of a second, for all the complete cycles. Then they determined how many of these lengths would fit into one second. Finally, they multiplied this value by the number of complete cycles in one of these lengths, thus arriving at the cycles per second, or the frequency, of their data. When I asked groups to explain their solutions to the class, the differences in their approaches were a rich source of discussion that engaged the students in mathematical reasoning and enhanced their understanding of mathematics.

Student groups use their approaches for finding the frequency of their data to fill or remove water from the note bottle until they find a sound wave with a frequency approximating that of their desired musical note. They learn that increasing the amount of water increases the frequency and that decreasing the amount of water decreases the frequency. The amount of water in the bottle determines the length of the air column in which the air molecules oscillate. The vibrations of the air column determine the note. As in wind instruments, the shorter the air column, the faster the vibrations, thus the higher the frequency of the note. These connections help students develop their understanding about the physics of sound production and its connections to mathematics.

When all the groups find their notes, the class plays "Mary Had a Little Lamb" or another song, following along with the sheet music on the overhead transparency. It usually takes several tries before they succeed! Then the groups share and compare the methods that they developed to create their musical notes, including the appropriateness and accuracy of their methods. Explaining their solutions and comparing them with other solutions helps the students develop a richer understanding of mathematics. Each student also writes an explanation of her or his group's solution (see **fig. 3** and **fig. 4**). Also important is the discussion that arises about the relationship between the frequency of the notes and the amount of water in the bottles.

### RELATED TASKS

Creating musical notes with the bottles can lead to other mathematics tasks. For example, the stu-

Okay, I just got home & am now looking at our data to feel comfortable with it. I have to own up to the fact that I thought our method was faulty, I looked at the window, from  $x$ -min to  $y$ -max was .025 seconds, however, when I looked at the graph & traced the line, I found that the viewable part of our graph was .00092231 to .02401889 sec or .02309658 sec Not .025.

Since in this interval there are exactly 9 periods, I know that if you divide 1 second by .02309658 sec my screen represents  $\frac{1}{.02309658}$  parts of a second. Multiplying 9 periods time  $\frac{1}{.02309658}$  we find we have 389.668087 periods per second which is about as close to 392 periods (which is the periods necessary for the note G) as we will get.

Fig. 4  
Student explanation of note using all complete cycles

*Students can investigate the ratios of the musical notes and compare the scales*

dents can measure and graph the height of the air column in the bottle for each note, that is, air column versus frequency, or air column versus note. They can use these graphs to describe the relationship between the notes and the air-column height and to predict the height of the air column needed for other notes. Students can also explore the height of the water and the volume of water needed to produce particular notes. The inverse relationship between the frequency and the period of their musical notes can be made explicit by asking the students to create a frequency-versus-period graph on the basis of data from all the musical notes created by the groups. By using the graph and the table of values, students can create an algebraic representation for the relationship, that is,

$$F = \frac{1}{p}.$$

Teachers can also develop other activities that promote connections between mathematics and music. For example, students can explore the math-

ematics related to the family of violins. Reinthaler (1990) suggests possible mathematical investigations related to string instruments and musical scales. Students can also explore the mathematics of harmonic motion, an important concept in the creation of music and musical instruments, as described by Ronan (1967).

### EXPLORING SINE FUNCTIONS

Creating a musical note allows students to use trigonometric functions to model and explain actual data. The periodicity and oscillatory nature of the sound-wave data can be approximated by a sine function, that is,  $y = a \sin b(x + c) + d$ , or a cosine function, if desired. For an initial introduction to sine functions, the **activity sheet** guides students' reasoning, problem solving, and development of understanding while they construct a sine function representing their sound wave. The task should be more open-ended for more advanced students.



*Explaining and comparing solutions helps students develop a rich understanding*

After the sound data are collected into a group's calculator, the data can be transferred to each group members' calculator using the [LINK] feature. Group members must have the same data to enable them to compare their work and help one another. Students can enter a generic sine function in the **Y=** menu, for example,  $Y_1 = A \sin B(X+C)+D$ . They can vary values for **A**, **B**, **C**, and **D** by storing the values directly into the variables on the calculator, that is, **2** [STO] **A**. Graphing the function produces immediate feedback about parameter changes.

The initial values stored in **A**, **B**, **C**, and **D** should be 1, 1, 0, 0, respectively, that is,  $y = \sin x$ . When the students press [GRAPH],  $y = \sin x$  does not appear on the screen with the sound-wave data. A closer look at the dimensions of the window for the sound data (see **fig. 5**) reveals that the sound data have a very small domain and range distinct from  $y = \sin x$ . It is difficult to showcase both sets of points in the same window. By experimenting with such different [WINDOW] settings as **Ymin = -1**, which is the minimum for  $y = \sin x$ , and **Xmax = 0.1**, a small  $x$ -value, students can discuss important differences between the shape of the sound data and that of the sine function.

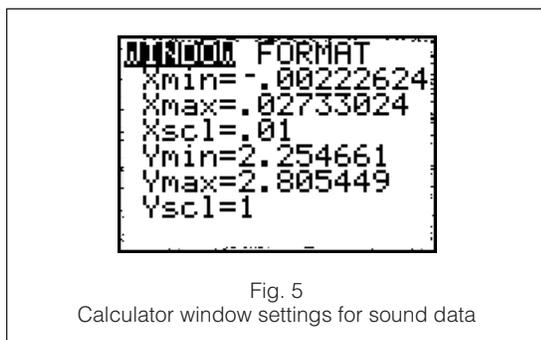


Fig. 5  
Calculator window settings for sound data

One important difference between the sound data and  $y = \sin x$  is the value of the vertical center. The graph of  $y = \sin x$ , that is,  $Y_1$ , is centered vertically on the line  $y = 0$ . The sound data, however, are centered vertically on a  $y$ -value above  $y = 0$ . Some students make a horizontal line on the graph with **2nd** [DRAW] **3**. This line can be moved up or down with the arrow keys to approximate the minimum and maximum  $y$ -values of the sound data. **Figure 6** shows the minimum  $y$ -value of the sound data in this article to be approximately 2.308. Similarly, the maximum value is approximately 2.744. The average of the minimum and maximum values produces an approximate value for the vertical center, or 2.526 in this example.

Storing the value for the vertical center in parameter **D**, that is, **2.526** [STO] **D**, shifts  $Y_1$  to a desired horizontal position, as shown in **figure 7**.

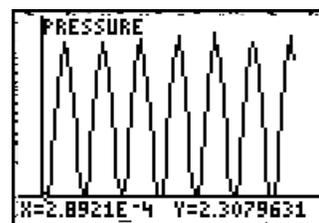


Fig. 6  
Horizontal line to approximate minimum  $y$ -value

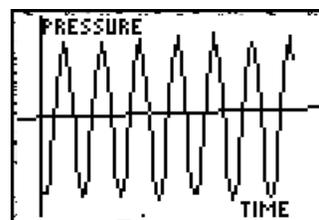


Fig. 7  
Translating  $Y_1$  to same line of vertical center as sound data

**Figure 8** presents two different student methods for determining the vertical center for the sound data, that is, parameter **D**. During small-group and whole-class discussions, students compared the appropriateness and accuracy of their methods. For example, they argued that the second method in **figure 8** is generally more accurate than the first, which involves a visual estimate. In class discussion, the students also realized that the vertical center for all the notes tended toward the same value, that is, 2.49, 2.491, 2.5, and so on. They conjectured that this value represented normal air pressure because the sound data are a succession of pressure variations above and below normal air pressure.

**Figure 7** reveals a distinct difference between the frequency of oscillations of the sound data and  $Y_1$ , for example,  $y = \sin x + 2.526$ . Parameter **B** in  $Y_1$  affects the frequency in the oscillations of the sine function, currently **B = 1**. When creating the musical notes, we determined the frequency of the sound-wave data. This frequency, however, is in cycles per one second, whereas the parameter **B** in  $Y_1$  measures cycles per  $2\pi$ . Thus,

$$B = \frac{\text{cycles}}{\text{sec}} \cdot 2\pi.$$

For **figure 7**, **B** is approximately 1712.809, that is,  $272.602 \cdot 2\pi$ . Storing 1712.809 in **B** produces a period for the graph of  $Y_1$ , that is,  $y = \sin 1712.809x + 2.526$ , that represents the period of the sound data. However, the amplitude of  $Y_1$ , currently **A = 1**, is too large. To determine the amplitude, or loudness, of

the sound data, students can find the distance from the line of vertical center for the sound data to the maximum height of the oscillations. For the sound data in **figure 7**, the amplitude is approximately 0.218. Storing this value in **A** produces **figure 9** where  $Y_1 = 0.218 \sin 1712.809x + 2.526$ .

**Figure 9** shows that  $Y_1$  needs a phase shift to accurately approximate the collected sound-wave data. This phase shift is needed because of the data-collection process rather than the musical note. Students can approximate the length of the phase shift by calculating the distance, along a horizontal line, between a point on the sound-wave data and a corresponding point on a cycle of  $Y_1$ . For the sound data in **figure 9**, the phase shift is 0.0022 to the left. Storing this value in parameter **C** produces the desired shift. **Figure 10** demonstrates the close fit of  $Y_1$  in approximating the collected sound data. The algebraic representation of the sound-wave data in the foregoing discussion is  $y = 0.218 \sin 1712.809(x + 0.0022) + 2.526$ . As students created the sine function that approximated their sound data, they enhanced their understanding of sine functions and sound.

The **activity sheet** guides students in creating a mathematical model representing a sound wave, engaging them in mathematical reasoning and communication. After determining the sinusoidal function for their musical note, students compare the mathematical models for different notes. They discuss the differences in their parameters in relation to the oscillations of their sound waves and the actual sound. For example, the values of the **B** parameter in  $Y_1$  are compared with respect to the frequency of the oscillations of their sound data and the pitch of their note. They also compare the values of the **A** parameter in relation to the amplitude of their sound wave and loudness of their sound and to the values of **D** as the point of equilibrium, that is, normal air pressure. Through their discussions, students further develop their understanding of the graphical and algebraic representations of sinusoidal functions and the creation of mathematical models of real-world data. Students can also create cosine functions that approximate their sound data and compare these functions with their previously created sine functions.

### REFLECTING ON THE TASKS

The mathematics tasks discussed here actively engaged students in doing mathematics. Using technology to achieve a class goal of playing a tune with water bottles promoted a positive attitude toward mathematics among the students. The context of these tasks sparked students' interest and facilitated collaboration and mathematical communication among the students. The rewards of this activity were many, including enhancing students'

$D = 2.491$  used the Trace Key.

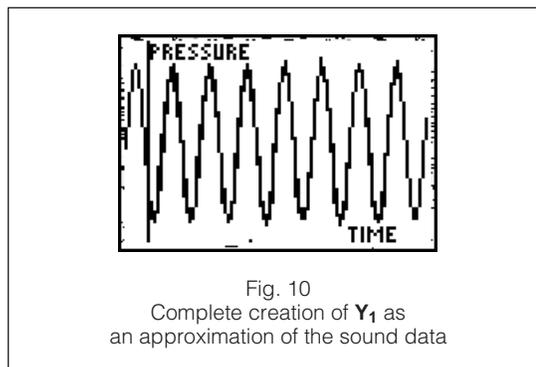
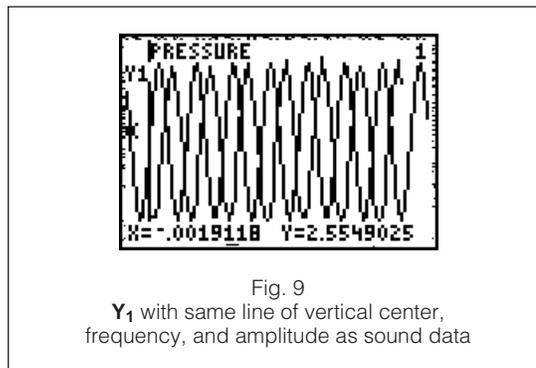
(a)  
Solution 1

$Y_1 = \sin x + 2.49$

I found the  $Y_{MAX}$  and  $Y_{MIN}$  of 1.5 by using the MATH MAX & MIN. Then I subtracted the MIN from the MAX, divided by 2 and added this # to the MIN.

(b)  
Solution 2

Fig. 8  
Two solutions for parameter **D**, activity sheet question 3(a)



understanding of mathematics and my own understanding of the physics of sound production.

### REFERENCES

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(Worksheets begin on page 96)

Playing a musical tune promoted a positive attitude

**Goal**

To create an algebraic mathematical model for a sound wave

**Materials**

- 1 Calculator-Based Laboratory (CBL)
- 1 CBL microphone
- 1 TI-82 graphing calculator with **TUNE** program for every pair of students
- 1 link cable
- 1 20-ounce plastic soft-drink bottle partially filled with water

**Collecting sound data**

1. Connect the CBL, CBL microphone, and TI-82 calculator. Execute the **TUNE** program to collect sound data produced by blowing across the top of the soft-drink bottle. **TUNE** collects data for 0.02475 seconds.
  - a) Sketch the graph of your sound data. Include time values, or x-values, for the beginning and end of your collected data. Use the calculator **TRACE** feature.
  - b) Record the frequency of the sound produced with the bottle.

**Creating the sine function**

2. Select the **Y=** menu. Enter  $Y_1 = A \sin B(X+C)+D$ , a standard sine function. Store initial values for the parameters in the calculator. Enter the following:

1 **STO▶** A; 1 **STO▶** B; 0 **STO▶** C; 0 **STO▶** D

- a) Record the current  $Y_1$  using actual values rather than variables.
- b) Why it is reasonable to begin with the initial values given when exploring the creation of a sine function representing the sound data?
- c) Why is the graph of  $Y_1$  not on the same screen as the sound data? Can you modify the domain and range of the **WINDOW** so that both are seen on the same screen? Explain. Explore **WINDOW** settings as needed.

**Determining the vertical shift**

3. Under the **ZOOM** menu select **ZoomStat** to return to the original screen setting for the sound data. Translate  $Y_1$  to the vertical position of the sound data by changing the value stored in **D**.
  - a) Determine the value of the line  $y = d$  on which the collected sound data are centered vertically, that is, the horizontal line midway between the oscillations. Explain how you determined this value.
  - b) Enter the value into **D** using the **STO▶** key. Record the current  $Y_1$  using actual values rather than variables.
  - c) Sketch the new graph screen, and explain the change in the graph of  $Y_1$ .

*Determining the period*

4. Change the period of  $Y_1$  by changing the value stored in **B**.
- Multiply the frequency in question 1(b) by  $2\pi$ .
  - Enter this value into **B** using the **STO▶** key. Record the current  $Y_1$  using actual values rather than variables.
  - Sketch the new graph screen, and explain the change in the graph of  $Y_1$ .
  - Explain why **B** is  $2\pi$  times the frequency in cycles per second. Hint: Compare one cycle of the sound wave with one cycle of  $y = \sin x$ .

*Determining the amplitude*

5. Change the amplitude, the vertical height, of  $Y_1$  by changing the value stored in **A**.
- Determine the height of the sound-wave data from the line of vertical center found in 3(a) to the maximum or minimum  $y$ -value of the oscillations. Explain how this amplitude is related to the amplitude of  $y = \sin x$ .
  - Enter the height into **A** using the **STO▶** key. Record the current  $Y_1$  using actual values rather than variables.
  - Sketch the new graph screen, and explain the change in the graph of  $Y_1$ .

*Determining the phase shift*

6. Change the phase shift of  $Y_1$  by changing the value of **C**.
- Determine the value that  $Y_1$  needs to be shifted to the right or the left to coincide with the collected sound data. Explain how you determined the value.
  - Enter the appropriate value, that is, positive or negative, into **C** using the **STO▶** key. Record the current  $Y_1$  using actual values rather than variables.
  - Sketch the new graph screen, and explain the change in the graph of  $Y_1$ .

$Y_1$  should be an algebraic model that represents the sound-wave data. If the graph of  $Y_1$  does not closely approximate the graph of the sound-wave data, check that  $Y_1$  is entered correctly, as in question 2. Then modify the values stored in the parameters **A**, **B**, **C**, or **D** as needed to better approximate the sound data. Record your final  $Y_1$ .